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On speciality of Jordan brackets

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Abstract. We conjecture a criterium of speciality for Jordan superalgebras of brackets that generalizes the corresponding criterium for Jordan Poisson superalgebras from [8]. We prove the necessarity condition of the criterium. The question on its sufficiency remains open.

Dedicated to Professor Miguel Ferrero
on occasion of his 70-th anniversary

1. Introduction

Let $\Gamma = \Gamma_0 + \Gamma_1$ be an associative commutative superalgebra over a ground field F, $char\, F \neq 2$, with a bracket $\{\ ,\}: \Gamma \times \Gamma \to \Gamma$. Consider the direct sum of two copies of the vector space Γ

$$J = \Gamma + \Gamma x,$$

with the product

$$\begin{aligned} a \cdot b &= ab, \\ a \cdot bx &= (ab)x, \\ (bx) \cdot a &= (-1)^{|a|}(ba)x, \\ ax \cdot bx &= (-1)^{|b|}\{a, b\}, \end{aligned}$$

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where $a, b \in \Gamma_0 \cup \Gamma_1$, juxtaposition stands for the product in Γ and $(-1)^{|a|} = (-1)^k$ if $a \in \Gamma_k$. We will refer to J as a *Kantor double* of $(\Gamma, \{\ ,\})$ and will denote $J = Kan(\Gamma, \{\ ,\})$.

A bracket $(\Gamma, \{\ ,\})$ is called Jordan if the Kantor double $Kan(\Gamma, \{\ ,\})$ is a Jordan superalgebra. I. L. Kantor [2] proved that every Poisson bracket is Jordan.

In [4] it was shown that all Jordan superalgebras that correspond to the so called "superconformal algebras" are Kantor doubles or are embeddable in Kantor doubles (the Cheng-Kac series). Moreover, the same is true for all simple finite-dimensional Jordan superalgebras over a field of prime characteristic with a nonsemisimple even part (see [7]).

A Jordan superalgebra J is called *special* if it can be embedded into a Jordan superalgebra of the type $A^{(+)}$, which is obtained from an associative superalgebra A via the new multiplication $a \cdot b = \frac{1}{2}(ab + (-1)^{|a||b|}ba)$. Furthermore, J is called *i-special* if it is a homomorphic image of a special one.

In [8] it was proved that for every Poisson bracket the corresponding Kantor double $Kan(\Gamma, \{\ ,\})$ is i-special. Moreover, in this case $Kan(\Gamma, \{\ ,\})$ is special if and only if $\{\{\Gamma, \Gamma\}, \Gamma\} = 0$.

In [6] it was proved that all Jordan brackets are embeddable into Poisson brackets. In view of [8] this implies that all Jordan Kantor doubles are i-special. The problem of speciality of certain Jordan brackets was also considered in [6], and it was proved, in particular, that the Jordan superalgebras of the Cheng-Kac series are special.

Here we return to the question of speciality of Jordan superalgebras of brackets in general case. We formulate a conjectural criterium of speciality for these superalgebras that generalizes the corresponding criterium from [8] for Jordan Poisson superalgebras. We prove that this condition is verified in special superalgebras and we conjecture that it implies the speciality as well. However, the conjecture remains open.

2. A necessary condition for speciality of Jordan brackets

We start with some definitions and notations.

By a superalgebra we mean a Z_2-graded algebra $A = A_0 + A_1$. Let $G = \langle 1, e_i, i \geq 1 \,|\, e_i e_j + e_j e_i = 0 \rangle$ denote the Grassmann (or exterior) algebra. Then $G = G_0 + G_1$ is a Z_2-graded algebra, where G_0, G_1 are linear spans of all tensors of even and odd length, respectively.

Let $\mathcal{V}$ be a variety of algebras defined by homogeneous identities (see [1, 10]). A superalgebra $A = A_0 + A_1$ is said to be a *$\mathcal{V}$-superalgebra* if its *Grassmann envelope* $G(A) = G_0 \otimes A_0 + G_1 \otimes A_1$ lies in $\mathcal{V}$.

In this way one can define Lie superalgebras (for characteristic different from 3[1]), Jordan superalgebras, etc. Clearly, associative superalgebras are just Z_2-graded associative algebras.

If A is an associative superalgebra then the vector space A with a new operation $a \cdot b = \frac{1}{2}(ab + (-1)^{|a||b|}ba)$ is a Jordan superalgebra which is denoted $A^{(+)}$. A Jordan superalgebra is said to be *special* if it is embeddable into a superalgebra of type $A^{(+)}$. Otherwise J is said to be *exceptional*. A Jordan superalgebra is *i-special* if it is a homomorphic image of a special Jordan superalgebra.

Let $\Gamma = \Gamma_0 + \Gamma_1$ be a unital associative commutative superalgebra with a skewsymmetric bilinear superform $\{\ ,\} : \Gamma \times \Gamma \to \Gamma$, $\{\Gamma_i, \Gamma_j\} \subseteq \Gamma_{i+j}$ which we call *a bracket*.

Consider the Kantor double $Kan(\Gamma, \{,\}) = \Gamma + \Gamma x$ (see the Introduction), with the Z_2-grading given by $Kan(\Gamma, \{,\})_0 = \Gamma_0 + \Gamma_1 x$, $Kan(\Gamma, \{,\})_1 = \Gamma_1 + \Gamma_0 x$.

A bracket $\{\ ,\}$ is said to be *Jordan* if the Kantor double $Kan(\Gamma, \{,\})$ is a Jordan superalgebra. For a Jordan bracket $\{\ ,\}$, the mapping $D : a \mapsto \{a, 1\}$ is a derivation of Γ (see [5]). Moreover, in [5, 3] it was proved that a bracket $\{\ ,\}$ is Jordan if and only if for arbitrary elements $a, b, c \in \Gamma_0 \cup \Gamma_1,\ x \in \Gamma_1$ hold the identities

$$\begin{aligned}
\{a, bc\} &= \{a, b\}c + (-1)^{|a||b|}b\{a, c\} - D(a)bc, \qquad (1)\\
J(a, b, c) : &= \{\{a, b\}, c\} + (-1)^{|a||b|+|a||c|}\{\{b, c\}, a\} \\
&\qquad + (-1)^{|a||c|+|b||c|}\{\{c, a\}, b\} \\
&= -\{a, b\}D(c) - (-1)^{|a||b|+|a||c|}\{b, c\}D(a) \\
&\qquad - (-1)^{|a||c|+|b||c|}\{c, a\}D(b), \qquad (2)\\
\{\{x, x\}, x\} &= -\{x, x\}D(x). \qquad (3)
\end{aligned}$$

Identity (3) is needed only in characteristic 3 case, otherwise it follows from (2) (see [3]).

A Jordan bracket $\{\ ,\}$ is called a *Poisson bracket* if $D(a) = 0$ for any $a \in \Gamma$. Observe that in this case Γ is a Lie superalgebra with respect to the bracket, and the identities (1) and (2) transform to the graded *Leibniz* and *Jacobi* identities.

A Jordan bracket $\{\ ,\}$ is called a *vector bracket* if $\{a, b\} = D(a)b - aD(b)$ for any $a, b \in \Gamma$.

Let us call a Jordan bracket special (i-special) if the corresponding Kantor double is special (respectively, i-special). We first formulate some previous results on speciality of Jordan brackets:

[1]In characteristic 3 case the definition of a Lie superalgebra needs an aditional condition: $[[x, x], x] = 0$ for any odd element x.

Theorem 2.1. [5, 9] *Every vector bracket is special.*

Theorem 2.2. [8] *Every Poisson bracket* $\{\ ,\}$ *is i-special. It is special if and only if* $\{\{\Gamma,\Gamma\},\Gamma\}=0$.

Theorem 2.3. [6] *For every Jordan bracket* $(\Gamma,\{\ ,\})$, *the corresponding Kantor double* $Kan(\Gamma,\{\ ,\})$ *is embedded into a superalgebra of Poisson bracket and therefore is i-special.*

Let $(\Gamma,\{\ ,\})$ be a Jordan bracket, $D : a \mapsto \{a,1\}$ be the associated derivation of Γ. Following [5], define a new bracket on Γ:

$$\langle a,b\rangle = \{a,b\} - D(a)b + aD(b).$$

If the bracket $\{\ ,\}$ is Poisson then $\langle\ ,\rangle = \{\ ,\}$, and if $\{\ ,\}$ is a vector bracket, then the corresponding bracket $\langle\, ,\rangle$ is trivial.

The main result of this paper is the following

Theorem 2.4. *If a Jordan bracket* $(\Gamma,\{\ ,\})$ *is special then the bracket* $\langle a,b\rangle$ *satisfies the identity*

$$\langle\langle a,b\rangle,c\rangle = -D(a)\langle b,c\rangle + (-1)^{|a||b|}D(b)\langle a,c\rangle. \tag{4}$$

Observe that identity (4) holds trivially for vector brackets, and for Poisson brackets it coincides with the condition $\{\{\Gamma,\Gamma\},\Gamma\}=0$ from [8].

We will need an auxiliary lemma. For superalgebras A,B we denote by $A\tilde{\otimes}B$ their graded tensor product, which coincides as a vector space with the usual tensor product $A\otimes B$ and has the following gradation and multiplication:

$$(A\tilde{\otimes}B)_0 = A_0\otimes B_0 + A_1\otimes B_1,\ (A\tilde{\otimes}B)_1 = A_0\otimes B_1 + A_1\otimes B_0,$$
$$(a\tilde{\otimes}b)(c\tilde{\otimes}d) = (-1)^{|b||c|}ac\tilde{\otimes}bd.$$

It is easy to see that a graded tensor product of two commutative (associative) superalgebras is commutative (associative) as well.

Lemma 2.5. *Let* Λ *be an associative commutative superalgebra.*

i) If a Jordan superalgebra J *is special, then the graded tensor product* $\Lambda\tilde{\otimes}J$ *is a special Jordan superalgebra as well.*

ii) For a Jordan bracket $(\Gamma,\{\ ,\})$, *the following isomorphism holds:*

$$\Lambda\tilde{\otimes}Kan(\Gamma,\{\ ,\}) \cong Kan(\Lambda\tilde{\otimes}\Gamma,\{\ ,\}),$$

where the bracket on the commutative superalgebra $\Lambda\tilde{\otimes}\Gamma$ *is defined via*

$$\{\lambda\tilde{\otimes}a,\mu\tilde{\otimes}b\} = (-1)^{|a||\mu|}\lambda\mu\tilde{\otimes}\{a,b\}.$$

Proof. i). Let A be an associative superalgebra and $i : J \to A^{(+)}$ be a monomorfism of Jordan superalgebras. Then one can easily check that the mapping $\lambda\tilde{\otimes}a \mapsto \lambda\tilde{\otimes}i(a)$ is a monomorfism of a superalgebra $\Lambda\tilde{\otimes}J$ into the superalgebra $(\Lambda\tilde{\otimes}A)^{(+)}$. It is clear that $\Lambda\tilde{\otimes}A$ is associative, hence $\Lambda\tilde{\otimes}J$ is a special Jordan superalgebra.

ii). The direct calculation shows that the mapping $\lambda\tilde{\otimes}(a+bx) \mapsto \lambda\tilde{\otimes}a + (\lambda\tilde{\otimes}b)x$ is an isomorphism of the superalgebras $\Lambda\tilde{\otimes}Kan(\Gamma,\{\ ,\}) = \Lambda\tilde{\otimes}(\Gamma+\Gamma x)$ and $Kan(\Lambda\tilde{\otimes}\Gamma,\{\ ,\}) = \Lambda\tilde{\otimes}\Gamma + (\Lambda\tilde{\otimes}\Gamma)x$. □

Corollary 2.6. *If a Jordan bracket* $(\Gamma,\{\ ,\})$ *is special then so is the bracket* $(G(\Gamma),\{\ ,\})$*, where* $G(\Gamma)$ *is the Grassmann envelope of* Γ*, with the bracket* $\{\eta\otimes a, \theta\otimes b\} = \eta\theta\otimes\{a,b\}$.

Proof. In fact, if $\sqrt{-1}\in F$ then it is easy to see that $G(\Gamma)\cong(G\tilde{\otimes}\Gamma)_0$, and since $Kan((G\tilde{\otimes}\Gamma)_0,\{\ ,\}) \subseteq Kan(G\tilde{\otimes}\Gamma,\{\ ,\}) \cong G\tilde{\otimes}Kan(\Gamma,\{\ ,\})$, the speciality of $(\Gamma,\{\ ,\})$ implies that of $(G(\Gamma),\{\ ,\})$. If $\sqrt{-1}\notin F$, we can extend F and consider the corresponding scalar extension of Γ which is evidently special. □

Proof of the theorem. Observe first that a commutative associative superalgebra Γ with a superanticommutative bracket $\langle,\rangle$ satisfies identity (4) if and only if the Grassmann envelope $G(\Gamma)$, with the bracket $\langle\eta\otimes a,\theta\otimes b\rangle = \eta\theta\otimes(\{a,b\} - D(a)b + aD(b))$ and the derivation $D(\eta\otimes a) = \eta\otimes D(a)$, satisfies the identity

$$\langle\langle a,b\rangle,c\rangle = -D(a)\langle b,c\rangle + D(b)\langle a,c\rangle. \tag{5}$$

So, passing to $G(\Gamma)$, we see that in view of the Corollary it is sufficient to prove identity (5) for the case $\Gamma=\Gamma_0$.

Assume that $J = Kan(\Gamma,\{\ ,\})$ is special, that is, there exists an associative superalgebra $A = A_0 + A_1$ such that $J\subseteq A^{(+)}$. Below, all the elements are assumed to be homogeneous. We will identify elements of J with their images in A, and will denote, during the proof, by $a\cdot b$ and ab the products in J and in A respectively. Moreover, for $r,s\in A$ denote $[r,s] = rs - sr$, $r\circ s = rs + sr$. Without loss of generality we may assume that the superalgebra A is generated by the set $\Gamma\cup x$. We will need the following identities which are true in any associative algebra (see [8]):

$$[[a,b],c] = (b\circ c)\circ a - b\circ(c\circ a) = 4((b\cdot c)\cdot a - b\cdot(c\cdot a)), \tag{6}$$
$$[a\circ c, b\circ c] = [a,c^2]\circ b - [b,c^2]\circ a + ([a,b]\circ c)\circ c + [[a,c],[b,c]]. \tag{7}$$

Let $a,b,c\in\Gamma$, then we have

$$[a\cdot x, b\cdot x] = 2(a\cdot x)\cdot(b\cdot x) = 2\{a,b\}, \tag{8}$$

$$\begin{aligned}[a, x^2] &= [a \circ x, x] = 2[a \cdot x, x] = 2[a \cdot x, 1 \cdot x] \\ &= 4\{a, 1\} = 4D(a), \end{aligned} \tag{9}$$

and by (6)

$$[[a, b], x] = [[a, x], b] = [[a, b], c] = 0. \tag{10}$$

Now by (8), (7) and (9)

$$8\{a, b\} = [a \circ x, b \circ x] = 8D(a)b - 8D(b)a + 4[a, b]x^2 + [[a, x], [b, x]], \tag{11}$$

which yields

$$\begin{aligned}\langle a, b\rangle &= \frac{1}{2}[a, b]x^2 + \frac{1}{8}[[a, x], [b, x]], \\ \langle\langle a, c\rangle, b\rangle &= \frac{1}{2}[\langle a, c\rangle, b]x^2 + \frac{1}{8}[[\langle a, c\rangle, x], [b, x]].\end{aligned}$$

Consider

$$\begin{aligned}[\langle a, c\rangle, b] &= \frac{1}{2}[[a, c]x^2, b] + \frac{1}{8}[[[a, x], [c, x]], b] \\ &\overset{(10)}{=} \frac{1}{2}[a, c][x^2, b] + \frac{1}{8}([[[a, x], b], [c, x]] + [[a, x], [[c, x], b]]) \\ &\overset{(9),(10)}{=} -2[a, c]D(b);\end{aligned}$$

$$\begin{aligned}[\langle a, c\rangle, x] &= \frac{1}{2}[[a, c]x^2, x] + \frac{1}{8}[[[a, x], [c, x]], x] \overset{(10)}{=} \frac{1}{8}[[[a, x], [c, x]], x] \\ &\overset{(6)}{=} \frac{1}{8}((([c, x] \circ x) \circ [a, x] - [c, x] \circ (x \circ [a, x])) \\ &= \frac{1}{8}([c, x^2] \circ [a, x] - [c, x] \circ [a, x^2]) \\ &\overset{(9)}{=} \frac{1}{2}(D(c) \circ [a, x] - D(a) \circ [c, x]) = D(c)[a, x] - D(a)[c, x].\end{aligned}$$

Furthermore,

$$\begin{aligned}[[\langle a, c\rangle, x], [b, x]] &= [D(c)[a, x] - D(a)[c, x], [b, x]] \\ &\overset{(10)}{=} D(c)[[a, x], [b, x]] - D(a)[[c, x], [b, x]] \\ &\overset{(11)}{=} D(c)(8\{a, b\} - 8D(a)b + 8D(b)a - 4[a, b]x^2) \\ &\quad -D(a)(8\{c, b\} - 8D(c)b + 8D(b)c - 4[c, b]x^2) \\ &= 8(D(c)\langle a, b\rangle - D(a)\langle c, b\rangle \\ &\quad -\frac{1}{2}(D(c)[a, b] - D(a)[c, b])x^2).\end{aligned}$$

Therefore,

$$\begin{aligned}\langle\langle a,c\rangle,b\rangle &= -[a,c]D(b)x^2 + D(c)\langle a,b\rangle - D(a)\langle c,b\rangle \\ &\quad -\frac{1}{2}(D(c)[a,b]-D(a)[c,b])x^2 \\ &= D(c)\langle a,b\rangle - D(a)\langle c,b\rangle \\ &\quad -\frac{1}{2}(2D(b)[a,c]+D(c)[a,b]-D(a)[c,b])x^2.\end{aligned}$$

Consider now

$$\begin{aligned}4D(a)\circ[a,b] &\overset{(9)}{=} [a,x^2]\circ[a,b] = [a\circ[a,b],x^2] - a\circ[[a,b],x^2] \\ &= [[a^2,b],x^2] - a\circ([[a,b],x]\circ x) \overset{(10)}{=} [[a^2,b],x]\circ x.\end{aligned}$$

But $a^2 = a\cdot a \in \Gamma$, hence $[[a^2,b],x] = 0$ by (10) and $D(a)\circ[a,b] = 2D(a)[a,b] = 0$. Linearizing the last identity, we get

$$D(a)[c,b] + D(c)[a,b] = 0,$$

which implies

$$\begin{aligned}&2D(b)[a,c] + D(c)[a,b] - D(a)[c,b] \\ =\ &(D(b)[a,c]+D(c)[a,b]) + (D(a)[b,c]+D(b)[a,c]) = 0.\end{aligned}$$

Therefore,

$$\langle\langle a,c\rangle,b\rangle = D(c)\langle a,b\rangle - D(a)\langle c,b\rangle,$$

which proves the theorem. □

It remains an open question, whether the identity (4) is sufficient for a Jordan bracket to be special.

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